*Journal of Statistical Physics, Vol. 69, Nos. 3/4, 1992*

# Phase Structure of Two-Dimensional Spin Models and Percolation

**A. Patrascioiu**[1] **and E. Seiler**[2]

*Received November 13, 1991; final May 15, 1992*

For a class of classical spin models in 2D satisfying a certain continuity constraint it is proven that some of their correlations do not decay exponentially. The class contains discrete and continuous spin systems with Abelian and non-Abelian symmetry groups. For the discrete models our results imply that they show either long-range order or are in a soft phase characterized by powerlike decay of correlations; for the continuous models only the second possibility exists. The continuous models include a version of the plane rotator $[O(2)]$ model; for this model we rederive, modulo two conjectures, the Fröhlich–Spencer result on the existence of the Kosterlitz–Thouless phase in a very simple way. The proof is based on percolation-theoretic and topological arguments.

**KEY WORDS:** Percolation; classical ferromagnets.

## 1. INTRODUCTION

Percolation models are the simplest models of statistical mechanics showing a nontrivial phase structure. On the other hand, since the groundbreaking work of Fortuin and Kasteleyn,[(1)] a rigorous connection has been known between some spin models (the Potts models) and certain models of correlated percolation. This idea can be extended to a wider class of models by embedding Ising variables in them, as first noted in the papers by Brower and Tamayo[(2)] and Patrascioiu[(3)] in connection with a new Monte Carlo updating scheme for the $O(N)$ classical ferromagnets.

[1] Physics Department and Center for the Study of Complex Systems, University of Arizona, Tucson, Arizona 85721.

[2] Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, P.O. Box 40 12 12, Munich, Germany.



0022-4715/92/1100-0573$06.50/0 

In this paper we use these ideas to prove the absence of exponential clustering in certain two-dimensional ferromagnetic systems. These models include discrete spin systems with Abelian and non-Abelian symmetry as well as a version of the $O(2)$ (plane rotator) model; we thus rederive the celebrated results of Fröhlich and Spencer[4] on the existence of the Kosterlitz–Thouless phase for this model. In a separate paper one of us[5] proposes an extension of these arguments to the non-Abelian $O(N)$ models $(N \geq 3)$.

Our arguments described here depend on some conjectures: One of them is essentially an extension of the Mermin–Wagner theorem to our models with continuous symmetry; the other one is a generalization of Harris's result[6] on the noncoexistence of disjoint percolating clusters in two dimensions. A precise formulation of our conjectures is given in Section 3. We have not been able to prove them, but we do not think anybody will doubt their truth. Under the assumption of these conjectures our arguments are rigorous. A short announcement of our results appears in ref. 7.

## 2. THE MODELS

We consider two-dimensional classical spin models with nearest neighbor interaction on the triangular lattice $T$. The spins always take values in a unit sphere $S_{N-1}$ or a discrete subset of it, such a regular polytope $P$. For any finite subset $\Lambda \subset T$ the model is specified by a probability measure of the form

$$d\mu_\Lambda = Z_\Lambda^{-1} \chi_\Lambda^\varepsilon(\{S(x)\}) \exp[-\beta H_\Lambda(\{S(x)\}] \prod_x dS(x) \tag{1}$$

where $H_\Lambda$ is a ferromagnetic nearest neighbor interaction, typically the standard nearest neighbor action (s.n.n.a.)

$$H = -\sum_{\langle xy \rangle} S(x) \cdot S(y) \tag{2}$$

$\chi_\Lambda^\varepsilon$ is a constraint enforcing that n.n. spins only differ by at most a certain $\varepsilon > 0$; in other words $\chi_\Lambda^\varepsilon$ is the characteristic function of the set of spin configurations satisfying

$$|S(x) - S(y)| \leq \varepsilon \tag{3}$$

for all n.n. pairs $x, y$. So this constraint imposes a certain Lipschitz continuity on the configurations. It is clear that the constrained model can be obtained as a limit of models with bounded n.n. coupling. At low temperatures ($\beta \gg \varepsilon^{-2}$) the constraint will be unimportant because the s.n.n.a. by itself will make configurations violating it very improbable. Two rigorous results make this intuitively obvious statement more precise:

**Proposition 1** (Georgii[8]). In the s.n.n.a. model defined by (1) but without the constraint $\chi_\Lambda^\varepsilon$, for any $\varepsilon > 0$ at sufficiently low temperature there is a percolating cluster of bonds satisfying $|S(x) - S(y)| < \varepsilon$.

**Proposition 2** (Bricmont and Fontaine[9]). In the s.n.n.a model (without constraint) for any $\varepsilon > 0$ at sufficiently large $\beta$ the probability of having a set of $C$ bonds satisfying $|S(x) - S(y)| \geqslant \varepsilon$ is bounded by $\exp(-\beta a C \varepsilon^2)$ for some $a > 0$.

These results suggest strongly that at large $\beta$ the constraint embodied in $\chi_\Lambda^\varepsilon$ does not affect the behavior of the model in any essential way, provided $\varepsilon \ll (1/\beta)^{1/2}$. It should also be noted that introduction of the constraint does not affect the perturbative expansion in powers of $1/\beta$ whenever it exists (for instance, in a finite volume). But it does eliminate the possibility of using a conventional high-temperature expansion, because the constraint always enforces at least short-range order. This suggests that the constraint models might not have a high-temperature phase with exponential clustering, a fact that will be established for subset of these models in this paper. These models are therefore interesting in their own right. But we introduced the constraint mainly for the sake of convenience, to avoid the technical complications arising from the need to introduce estimates involving the exceptional sets ("defects") where it is violated. Strictly speaking, we are not dealing with Gibbs states, but "specifications" in the terminology of Georgii[10]; as remarked, however, they can be obtained as limits of Gibbs states by sending certain parameters in the Hamiltonian (action) to $\infty$. We will nevertheless talk about "Gibbs" states by a slight abuse of language.

Finally, the *a priori* mesure $dS$ will for the continuous models be either the standard $O(N)$ invariant spherical measure on $S_{N-1}$ or a measure invariant under an $O(2)$ subgroup of $O(N)$, such as the invariant measure multiplied by the characteristic function of the set $\{S \mid S \cdot n < c\}$; for the discrete models it will be a measure concentrated on the vertices of a polytope $P$ and invariant under a discrete symmetry group $H \subset O(N)$. The polytope $P$ need not really be regular; it is sufficient that the symmetry group $H$ acts transitively on its vertices. A nonregular example is the "truncated icosahedron" discussed in Section 5.

## 3. FACTS AND CONJECTURES

In this section we recall certain known facts relevant for our analysis and state some conjectures that are very plausible but unproven extensions of those facts; our main results will be derived under the assumption of (one or two of) those conjectures.

### 3.1. Ergodicity

The Mermin–Wagner theorem states in short that for 2D models, invariant under a compact connected Lie group $G$ and with short-range interactions, all Gibbs states will also be invariant under $G$.

This theorem has been proven in various ways and under slightly different technical assumptions by various authors.[11–14] One of the technical restrictions needed in all the proofs is that the Hamiltonian be twice differentiable, which is violated in our Lipschitz continuous models. But it is plausible that the theorem remains valid for those models, because intuitively the symmetry is restored by "soft" spin waves that do not violate the constraint. In the Appendix we give a proof of this conjecture for a simplified version of such models with even stronger restrictions on the spin fluctuations.

The Mermin–Wagner theorem suggests something even stronger, namely that there is a unique translation-invariant infinite-volume Gibbs state specified by the action (Hamiltonian) (1). Bricmont *et al.*[15] (see also ref. 16) proved this for the s.n.n.a. $O(2)$ model. By the Birkhoff ergodic theorem this pure phase is then ergodic for the translation group, i.e., with probability 1 the space average of any observable (bounded local function of the spins) is equal to its expectation value in the Gibbs measure. Because of this uniqueness this state will then be automatically $G$-invariant (the symmetry group transforms translation-invariant Gibbs states into translation-invariant Gibbs states), so that the Mermin–Wagner theorem is implied by ergodicity. In the sequel we will need the following for the continuous models specified by (1):

**Conjecture 1** (Ergodicity). Any translation-invariant infinite-volume "Gibbs" state for the $O(N)$ models specified by (1) is ergodic for the lattice translation group.

For the discrete models Conjecture 1 cannot hold in general because there will be spontaneous symmetry breaking at least for large $\beta$. Therefore our results for discrete models are weaker: while for the $O(2)$ model we establish masslessness, for the discrete models we only obtain the absence of the high-temperature phase with exponential clustering, but we cannot decide by our method if they show long-range order or algebraic decay of correlations. For this weaker result, however, we do not need a replacement of Conjecture 1; it suffices to make use of the ergodic decomposition and work in an ergodic component.[10]

Ergodicity means that we can discover all properties of the infinite-volume Gibbs state by looking at a single "typical" configuration (the word "configuration" simply stands for a map $S$: $\mathbb{Z}^2 \to S_{N-1}$). Because of the Mermin–Wagner property this implies also that for the $O(N)$ models any "typical" configuration has to be $O(N)$ symmetric by itself in the sense that space averages are $O(N)$ symmetric; furthermore, any property depending on the choice of a unit vector $n \in S_{N-1}$ that holds for space averages in a typical configuration has to hold also in the same configuration with respect to any other unit vector. As an example, one may think of the average size of the "hemispherical" clusters defined by the connected components of $S^{-1}$ $(\{S \cdot n > 0\})$ that will play an important role in the following discussion.

To illustrate what is meant by a "typical" configuration, consider the following concrete realization: It can be proven that certain stochastic processes converge to the correct Gibbs state. Monte Carlo simulations can be thought of as realizations of such processes. After a certain number of sweeps of the lattice needed to achieve thermalization, one starts taking measurements essentially as time averages of the desired quantity. To measure $\langle A \rangle$, where $A$ is a local observable, one will compute the average value of this quantity over a large number of configurations produced by the Monte Carlo procedure. Ergodicity says that one could as well obtain the answer by running the Monte Carlo algorithm on a very large lattice until achieving thermalization, then in one given configuration compute

$$\frac{1}{|\Lambda|} \sum_{y \in \Lambda} A(y) \tag{4}$$

where $A(y)$ is the observable $A$ shifted by the lattice vector $y$. In the infinite-volume limit (4) will coincide with $\langle A \rangle$. This is the basis of the well-known procedure of spatial averaging used in numerical studies to improve the statistics.

There is, however, another consequence of ergodicity which will be useful: Let us assume that the dynamical variables of the system are divided into two groups; for instance, in the $O(N)$ model the components of the spins $S(x)$ parallel to a certain reference vector $n$, called $S_{||}$, and the transverse components $S_{\perp}$ perpendicular to $n$. Let us also consider an observable $A$ depending only on $S_{\perp}$. Clearly the expectation value of $A$ can be obtained by first taking the conditional expectation conditioned on the variables $S_{||}$ and then integrating over the variables $S_{||}$ (annealed prescription). By the assumed ergodicity we can replace the last integration by a space average in a typical configuration. In the context of Monte Carlo simulations this means that we can measure the expectation value of $A$ by first doing enough Monte Carlo sweeps to produce a well-thermalized configuration for the full system, then update only the variables $S_\perp$, measure $A$ and its translates, and finally average over the translates. In other words, we can look at the system, as far as the variables $S_\perp$ are concerned, as a system with suitable (quenched) random couplings. We should like to note that this use of ergodicity allows us to avoid the consideration of possibly non-Gibbsian measures which might be produced by integrating out the variables $S_{\|}(x)$; the price to be paid is the loss of translation invariance. However, the measure governing the distribution of the variables $S_{\|}$ and hence the random couplings of the $S_\perp$ system is translation invariant if the Gibbs state of the full model is, as we assume.

### 3.2. Impossibility of Simultaneous Percolation of Disjoint Clusters in Two Dimensions

Consider some site percolation problem in two dimensions. Suppose the underlying measure enjoys: (a) the symmetries of the lattice (translations and rotations), (b) the FKG property, and (c) the one-step Markov property.

Several rigorous papers[17–19] have used Harris's strategy[6] to prove that under these circumstances, in two dimensions, clusters of occupied sites and clusters of unoccupied sites cannot percolate at the same time. The basic reason why this is so in two dimensions is that once the origin has nonvanishing probability to be connected to infinity via a set of occupied sites, there is a nonvanishing probability for the origin to be surrounded by a closed circuit of such sites. The presence of such circuits prevents the unoccupied sites from percolating.

We would like to conjecture that this result holds under much more general circumstances, because the basic geometric reason for it is the "intuitively obvious" fact that in two dimensions percolating clusters produced by a measure invariant under translations and lattice rotations will get in each other's way. Specifically, we conjecture that for the $O(N)$ models considered here the following is true:

**Conjecture 2.** Consider a partition of the sphere $S_{N-1}$ into a set $B$ and its complement $\sim B$, where $B$ and $\sim B$ are both assumed to have nonempty interior. Then in two dimensions in a Gibbs measure for a nearest neighbor interaction obeying symmetry under lattice translations and rotations there is a.s. no simultaneous percolation of the clusters of $S^{-1}(B)$ and of $S^{-1}(\sim B)$.

For the discrete models also discussed here we have to give a slightly different formulation:

**Conjecture 2′.** Consider a nontrivial partition of the vertex set of the polytope $PM$ into a set $B$ and its complement $\sim B$. Then in two dimensions for a Gibbs measure corresponding to a nearest neighbor interaction symmetric under lattice translations and rotations there is no simultaneous percolation of the clusters of $S^{-1}(B)$ and of $S^{-1}(\sim B)$.

The missing ingredients in proving these conjectures are the FKG property and the one-step Markov property for the induced percolation measure, which are used in the proof of the analogous property for the Ising model.

### 3.3. Russo's Theorem on the Divergence of the Mean Cluster Size

Russo (ref. 19, Proposition 1) proved the following simple but important result:

**Proposition 3.** If in a translation-invariant site percolation problem the occupied sites do not percolate and the expected size of the cluster of occupied sites attached to the origin is finite, then the empty sites * percolate.

We will call this expected size of the cluster attached to the origin in short the mean cluster size. A logically equivalent formulation of this result is:[19] If in a translation-invariant percolation problem there is neither percolation of occupied sites nor * percolation of empty sites, then the expected size of the clusters of occupied sites as well as the * clusters of empty sites diverges.

For the triangular lattice we are considering, percolation and * percolation are the same, hence Russo's result adapted to the $T$ lattice gives the following result:

**Proposition 3′.** If in a translation-invariant percolation model on the $T$ lattice neither the empty nor the occupied sites percolate, the mean cluster size of both of them diverges.

## 4. EMBEDDED ISING VARIABLES AND INDUCED PERCOLATION MODEL

Our analysis is based upon the representation of our models as Ising models with random couplings; this is also the basis of the cluster Monte Carlo algorithms developed for these systems in recent years, which have led to a remarkable reduction in critical slowing down.[20-22] Let us first describe the Fortuin–Kasteleyn (FK) representation for the s.n.n.a. $O(N)$ ferromagnets with the action

$$H = -\sum_{\langle xy \rangle} S(x) \cdot S(y) \tag{5}$$

Choosing some unit vector $n \in S_{N-1}$, an alternative representation of the associated Gibbs measure in a finite volume $\Lambda$ becomes possible, using as dynamical variables $\{\sigma_x = \mathrm{sgn}(S(x) \cdot n),\ |S_{\|}(x)|,\ S_{\perp}(x)\}$:

$$d\mu_\Lambda = Z_\Lambda^{-1} \exp\left\{\beta \sum_{\langle xy \rangle} [\sigma_x \sigma_y |S_{\|}(x)|\, |S_{\|}(y)| + S_{\perp}(x)\, S_{\perp}(y)]\right\} dS\, d\sigma \tag{6}$$

where $S_{\|} = S \cdot n$, $S_{\perp} = S - n(S \cdot n)$, and $dS\, d\sigma$ is the obvious *a priori* measure. In terms of the variables $\sigma$, the system is that of an Ising ferromagnet, amenable to the Fortuin–Kasteleyn (FK) representation[1] [we can ignore the exceptional situation that $S_{\|}(x) = 0$ for some $x$, making the corresponding Ising variable ill defined: this event obviously has probability 0 in a finite system; by ergodicity the probability vanishes also in the thermodynamic limit]. This representation leads to the following correlated bond percolation problem: Given a configuration of the Ising spins, a bond between two nearest neighbor spins of the lattice is activated only when they are equal, and then only with probability

$$p_{xy} = 1 - \exp[-2\beta |S_{\|}(x)\, S_{\|}(y)|] \tag{7}$$

If we introduce the bond occupation variables $\{n_{xy}\}$, $n_{xy} \in \{0, 1\}$, they are therefore distributed according to the following conditional probability, given a configuration of the spins:

$$P(\{n_{xy}\} \mid \{S(x)\}) = Z_\Lambda^{-1} \prod_{\langle xy \rangle} \{\delta_{\sigma_x \sigma_y}[n_{xy} p_{xy} + (1 - n_{xy})(1 - p_{xy})] + (1 - \delta_{\sigma_x \sigma_y})(1 - n_{xy})\} \tag{8}$$

On the other hand, the conditional probability distribution of the Ising variables, given the bond occupation numbers, is obtained by assigning to each cluster $C \pm 1$ with probability 1/2:

$$P(\{\sigma_x\} \mid \{n_{xy}\}) = \prod_C \left\{\tfrac{1}{2} \prod_{x \in C} (1 + \sigma_x) + \tfrac{1}{2} \prod_{x \in C} (1 - \sigma_x)\right\} \tag{9}$$

The joint distribution of all the variables is given by the probability density

$$P(\{n_{xy}\}, \{S(x)\}) = Z_A^{-1} \prod_{\langle xy \rangle} \{\delta_{\sigma_x \sigma_y}[n_{xy} p_{xy} + (1 - n_{xy})(1 - p_{xy})] + (1 - \delta_{\sigma_x \sigma_y})(1 - n_{xy})\} \prod_{\langle xy \rangle} e^{\beta S(x) \cdot S(x)} \tag{10}$$

and it is not hard to see that (10) reproduces the standard Gibbs measure for the spins after summing over the bond occupation numbers.

One version of the cluster algorithms consists essentially in alternating updates of the bond occupation numbers and the Ising spins according to the conditional probabilities (8) and (9), randomly selecting a new reference vector $n$ in between.

Equations (8)–(10) have obvious generalization to more general nearest neighbor action for the $O(N)$ models of the form

$$H\{S(x)\} = \sum_{\langle xy \rangle} h(S(x) \cdot S(y)) \tag{11}$$

The Ising variables can be introduced as before. Only the bond activation probabilities have to be adjusted: Denote by $\mathcal{R}_n S$ the reflected image of $S$:

$$\mathcal{R}_n S = S - 2n(S \cdot n) \tag{12}$$

Then (7) has to be replaced by

$$p_{xy} = 1 - \exp[\min\{0, A\}] \tag{13}$$

with $A = \beta[(h(S(x) \cdot S(y)) - h(\mathcal{R}_n S(x) \cdot S(y))]$, whereas the form of (8), (9) remains unaltered. Obviously (10) will have to be replaced by

$$P(\{n_{xy}\}, \{S(x)\}) = Z_A^{-1} \prod_{\langle xy \rangle} \{\delta_{\sigma_x \sigma_y}[n_{xy} p_{xy} + (1 - n_{xy})(1 - p_{xy})] + (1 - \delta_{\sigma_x \sigma_y})(1 - n_{xy})\} \prod_{\langle xy \rangle} e^{\beta h(S(x) \cdot S(x))} \tag{10'}$$

In general the induced Ising model will not necessarily have only ferromagnetic couplings, so the derivation of the FK representation given in ref. 1 cannot be carried over. Nevertheless it is easy to check that (10′) after summing over the bond occupation numbers still gives the right Gibbs measure.

For our models with a Lipschitz constraint, considering them as limits of models with bounded actions, one obtains simply $p_{xy}=1$ unless both $|S(x)-S(y)| \leqslant \varepsilon$ and $|\mathscr{R}_n S(x)-S(y)| \leqslant \varepsilon$; this means that only bonds near the equator can be unoccupied.

For the discrete models one proceeds in essentially the same way; it is only necessary to orient the polytope $P$ in such a way that the reflection $\mathscr{R}_n$ maps the vertex set of $P$ into itself. It is allowed that $\mathscr{R}_n$ has fixed points: in this case the corresponding Ising variables can be assigned arbitrarily because they are not coupled.

The percolation problem defined by (8) can be decomposed into two steps: First one forms "hemispherical" (H) clusters of parallel Ising spins considered as bond clusters (and again it does not matter whether we consider the hemispheres as open or closed); then one deletes from them randomly bonds with probability $1-p_{xy}$, thereby obtaining the "Fortuin–Kasteleyn" (FK) clusters. Fortuin and Kasteleyn showed already[1] that the magnetic susceptibility of the s.n.n.a. Ising is equal to the expected size of the FK cluster attached to the origin. An analogous relation holds also in our "embedded" Ising system: Denote by $\langle \mathrm{FK} \rangle$ the expected size (measured by the number of lattices sites in it) of the FK cluster attached to the origin (often called the mean size of the FK cluster for short); then we have

$$\chi_{\mathrm{Is}}=\frac{1}{|\Lambda|} \sum_{x, y \in \Lambda} \langle \sigma_x \sigma_y \rangle = \langle \mathrm{FK} \rangle \tag{14}$$

In (14) the expectation values are taken with the finite-volume probability measure (10′). The thermodynamic limit is then taken for the 2-point function and the corresponding connectivity function of the FK clusters; this way we obtain a version of (14) in the thermodynamic limit. If the two-point function $\langle \sigma_x \sigma_y \rangle$ decays exponentially (uniformly in the volume), $\chi_{\mathrm{Is}}$ will have a finite limit as $\Lambda \to T$. Conversely, if $\langle \mathrm{FK} \rangle = \chi_{\mathrm{Is}}$ diverges, there will be massless modes in the system and exponential clustering has to fail for some observables. By the assumed ergodicity, the infinite-volume limit of $\langle \mathrm{FK} \rangle$ can also be obtained as the average over $x$ of the size of the cluster attached to $x$ in a "typical configuration."

The proof that under certain conditions in fact $\langle \mathrm{FK} \rangle = \chi_{\mathrm{Is}}$ diverges will be given in the next sections and uses some topological properties. For now we note the following fact, which is an easy consequence of the simple form Russo's theorem takes on the $T$ lattice (Proposition 3′):

**Lemma 1.** Consider a Gibbs state of an $O(N)$, $N \geqslant 1$, model on the $T$ lattice that is invariant under lattice translations as well as the symmetry group $O(N)$. Assume that Conjecture 2 holds. If we denote by $\langle \mathrm{H}^+ \rangle$ ($\langle \mathrm{H}^- \rangle$) the expected size of the upper (lower) hemispherical cluster attached to the origin, then $\langle \mathrm{H}^+ \rangle = \langle \mathrm{H}^- \rangle = \infty$.

*Proof.* By the assumed $O(N)$ invariance and Conjecture 2 none of the hemispherical clusters can percolate. So the lemma follows from Proposition 3′. ▮

For the discrete models there is an analogous statement:

**Lemma 1′.** Consider a Gibbs state of a discrete model with spins taking values in a polytope $P \subset S_{N-1}$ with invariance group $H \subset O(N)$ on the $T$ lattice. Assume that $H$ contains a reflection $\mathscr{R}_n$ acting on $P$ without fixed points. Assume furthermore that the state is invariant under lattice translations and rotations as well as the symmetry group $H$ and that Conjecture 2 holds. If we denote by $\langle \mathrm{H}^+ \rangle$ ($\langle \mathrm{H}^- \rangle$) the expected size of the upper (lower) hemispherical cluster attached to the origin, then $\langle \mathrm{H}^+ \rangle = \langle \mathrm{H}^- \rangle = \infty$.

Next we use the fact that for a model obeying a Lipschitz condition of the form (2) the only bonds that may be deleted in going from the H to the FK clusters have to have spins within distance $\varepsilon$ of the equator at both ends. In other words, the FK clusters contain the clusters of the "reduced hemispheres"

$$\mathscr{H}_\delta^+ \equiv \{S \in S_{N-1} \mid S_{\|} > \delta\} \tag{15a}$$

$$\mathscr{H}_\delta^- \equiv \{S \in S_{N-1} \mid S_{\|} < -\delta\} \tag{15b}$$

for all $\delta > \varepsilon$, and if the clusters of $\mathscr{H}_\delta^+$ and $\mathscr{H}_\delta^-$ have divergent mean size, so will the FK clusters. This leads us to the following criterion for masslessness for the $O(N)$ models:

**Proposition 4.** Assume that Conjectures 1 and 2 hold. Let $\delta > \varepsilon$. If the inverse image under the map $S$ of the equatorial strip

$$\mathscr{S}_\delta = \{S \in S_{N-1} \mid |S \cdot n| < \delta\} \tag{16}$$

does not percolate, then the Ising susceptibility (14) diverges and the system does not cluster exponentially.

*Proof.* The complement of the strip (16) consists of the two disjoint reduced hemispheres

$$\mathscr{H}_\delta^+ \equiv \{S \in S_{N-1} \mid S_{\|} > \delta\} \tag{17a}$$

$$\mathscr{H}_\delta^- \equiv \{S \in S_{N-1} \mid S_{\|} < -\delta\} \tag{17b}$$

The set $S^{-1}(\mathscr{H}_\delta^+ \cup \mathscr{H}_\delta^-)$ cannot percolate, because if it did, by the Lipschitz condition one of its disjoint subsets $S^{-1}(\mathscr{H}_\delta^+)$ and $S^{-1}(\mathscr{H}_\delta^-)$ would have to percolate; symmetry would then require both of them to percolate, which would contradict Conjecture 2. Now, by Russo's theorem (Proposition 3′) this means that each of the two disjoint subsets has clusters of divergent mean size. By the remark made above this implies divergence of $\langle \mathrm{FK} \rangle$. ▮

This proposition therefore reduces the proof of masslessness for the $O(N)$ models given by (1) to the proof that certain equatorial strips do not percolate, or equivalently, that certain reduced hemispheres have divergent mean cluster size. In the next section we will use the topology of the circle to show that this actually happens. In ref. 5, arguments are given that also in the $O(N)$, $N \geqslant 3$, models equatorial strips as in Proposition 4 do not percolate.

For the discrete models a similar criterion can be established, but it would require the use of Conjecture 1, which, as said before, cannot be expected to hold in general because of the occurrence of spontaneous symmetry breaking.

## 5. A TOPOLOGICAL ARGUMENT

In this section we use topological properties of the circle and certain polytopes to establish some elementary percolation properties for Lipschitz continuous maps from the $T$ lattice into these spaces. In the next section these properties will be used to prove our results about 2D spin systems.

We first look at maps from the $T$ lattice to general graphs: Let $G$ be a finite graph, $V(G)$ its vertex set, and $f$ be a map from the triangular lattice $T$ into $V(G)$:

$$f\colon\quad T \to V(G) \tag{18}$$

$f$ is assumed to be continuous in the sense that n.n. points of the lattice are mapped either into the same vertex or into vertices of $G$ connected by an edge. The main consequence of the continuity of the map $f$ for us is that the image of a connected subset of $T$ will be connected (in the obvious sense).

Our goal is to establish percolation properties of the inverse image under $f$ of certain subsets of $V(G)$. There are two properties which are important:

**Definition 1.** We say that a subset $C \subset V(G)$ has the property (P) ("percolates") under $f$ iff $f^{-1}(C)$ has an infinite connected component. If the complement of a set $C$ percolates, we say that $C$ "forms islands."

**Definition 2.** We say that a subset $C \subset V(G)$ has the property (R) ("forms rings") under $f$ iff any finite subset $F \subset T$ is surrounded by a connected component of $f^{-1}(C)$.

*Remark 1.* Obviously, if (P) holds for a certain set, it will hold for one of its connected components.

*Remark 2.* Note that the properties (P) and (R) are not mutually exclusive. In fact, in the cases of interest, coming from Gibbs measures invariant under lattice translations and rotations, presumably percolating sets will always automatically satisfy property (R). The reason is that in two dimensions for configurations produced with such a measure it is in general impossible that two disjoint sets percolate (cf. Conjecture 2); therefore, if $C$ percolates, percolation of its complement will be blocked by circuits of $f^{-1}(C)$, i.e., property (R) holds.

We consider disjoint decompositions of $V(G)$ into two connected subsets

$$V(G) = V_+ \mathbin{\dot\cup} V_- \tag{19}$$

If $S \subset V(G)$, we define its boundary $\partial S$ to be the set of vertices of $S$ connected by an edge to the complement of $S$ and its interior $\operatorname{int} S$ to be $S \backslash \partial S$.

The following fact is almost obvious:

**Proposition 5.** Assume that $R$ and $S$ form islands (i.e., their complements percolate), and assume furthermore that $R$ and $S$ "do not touch," i.e., there is no edge between a vertex of $R$ and a vertex of $S$. Then also $R \cup S$ forms islands.

*Proof.* Assume on the contrary that the complement of $R \cup S$ does not percolate. Then any finite subset of the lattice $T$ is surrounded by a connected circuit of $f^{-1}(R \cup S)$. By continuity of $f$ the image of this connected set has to be connected; therefore it will be contained entirely within $R$ or $S$. But this means that the complement of $R$ or of $S$ does not percolate, in contradiction with the assumption. ∎

Next we state another simple fact:

**Proposition 6.** One of the following three statements holds:

(1) $V_+$ percolates.

(2) $V_-$ percolates.

(3) $\exists$ a connected component of $\partial V_+$ and a connected component of $\partial V_-$ forming rings.

*Proof.* If neither $V_+$ nor $V_-$ percolates, the percolation has to be blocked by rings of the respective complement, i.e., each finite subset $F \subset T$ is surrounded by a circuit of $f^{-1}(V_+)$ and $f^{-1}(V_-)$. By continuity the same is true for $\partial f^{-1}(V_+) \subset f^{-1}(\partial V_+)$ and $\partial f^{-1}(V_-) \subset f^{-1}(\partial V_-)$. Again by continuity each such circuit can only consist of points belonging to one connected component, so (3) holds. ∎

We now introduce a class of graphs for which the connected components of the boundary always consist of single vertices.

**Definition 3.** A triangular face of a graph $G$ is a triple $\{v_1, v_2, v_3\}$ of vertices which are pairwise connected by an edge of $G$.

Then the following holds:

**Proposition 7.** If $G$ has no triangular faces, $f$ is constant on the connected components of $\partial f^{-1}(S)$ for any subset $S \subset V(G)$.

*Proof.* It suffices to consider a connected component of $\partial f^{-1}(S)$ consisting of more than one point. Because $T$ is self-matching, the connected components of $\partial f^{-1}(S)$ consist of circuits, so if $x \in \partial f^{-1}(S)$, it will have a neighbor $y \in \partial f^{-1}(S)$ and $x$ and $y$ will have a common neighbor $z$ with $f(z) \notin S$. Assume now $f(x) \neq f(y)$. Continuity requires that $f(x)$, $f(y)$, $f(z)$ form a triangular face, which was excluded. So $f(x)$ has to be equal to $f(y)$. ∎

Combining Propositions 6 and 7, we obtain the following result (using the finiteness of $G$):

**Proposition 8.** If $G$ does not have triangular faces, then one of the following three statements holds:

(1) $V_+$ percolates.

(2) $V_-$ percolates.

(3) There is a vertex $v_+ \in V_+$ and a vertex $v_- \in V_-$ forming rings.

*Examples* for graphs without triangular faces are as follows:

(a) The cube $\mathscr{C}$, considered as a graph in the obvious way.

(b) The (regular) dodecahedron $\mathscr{D}$, considered as a graph in the obvious way.

(c) The "buckminsterfullerene"; this is a polyhedron with 60 vertices with the icosahedral group $Y$ as a symmetry group. It can be obtained by truncating the regular icosahedron and is the molecular structure of the famous $C_{60}$ molecule.

(d) $G = \mathbb{Z}_n$, $n \geqslant 4$, considered as a graph by placing a line between any two adjacent points.

(e) $G = \mathbb{Z}$, considered as a graph by placing a line between any two adjacent elements; this graph, being infinite, is strictly speaking outside the setting discussed here, but we can apply our arguments if we identify elements of $\mathbb{Z}$ modulo some integer $k$.

If the map $f$ is a "typical" configuration coming from a Gibbs measure enjoying some additional symmetry properties, we can say more. Let us assume that there is a group $H$ acting transitively on $V(G)$ and that $f$ comes from a Gibbs state that is $H$-symmetric. Then sets related by a symmetry transformation have the same percolation properties. Let furthermore $V_+$ and $V_-$ be of equal cardinality and related by a symmetry transformation. Then if one of the two sets $V_+$ and $V_-$ percolates, so does the other. Assuming Conjecture 2, this cannot happen, so by Proposition 8, each vertex $v \in V(g)$ will form rings. This situation will arise in all the examples given above if $f$ is a typical configuration produced by the "Gibbs" measure (1). Only in case (d) do we have to assume that $n$ is even in order to be able to split $\mathbb{Z}_n$ into two equal pieces.

Proposition 8 can also be applied to some continuous models in the following way:

Let $M$ be a Riemannian manifold with a distance function $d(\cdot, \cdot)$ and

$$f\colon \quad T \to M \tag{20}$$

be Lipschitz continuous with Lipschitz constant $\varepsilon$, i.e.,

$$d(f(x), f(y)) \leqslant \varepsilon |x-y| \tag{21}$$

Assume further that $M$ can be decomposed into mutually disjoint subsets $M_i$, $i=1, 2,..., N$,

$$M = \bigcup_{i=1}^{N} M_i \tag{22}$$

such that for any pair $i, j$ either

$$d(M_i, M_j) = 0 \tag{23}$$

or

$$d(M_i, M_j) > \varepsilon \tag{24}$$

In that case we can associate a graph $G(M)$ to the decomposition of $M$ by taking the subsets $M_i$, $i=1, 2,..., N$ as vertices and placing a line between $M_i$ and $M_j$ iff $i \neq j$ and $d(M_i, M_j)=0$. Then $f$ induces a continuous map

$$f_G\colon \quad T \to G(M) \tag{25}$$

of the kind discussed before.

The trouble is that in many cases the graph obtained this way will have triangular faces, because three sets of the decomposition will touch; so by our methods here we cannot learn much because Proposition 8 does not apply. This happens, for instance, if we divide the 2-sphere into patches produced by projecting the cube or the regular dodecahedron from the center onto $S_2$. If, on the other hand, we divide any $(N-1)$-sphere into an equatorial strip and two polar caps as discussed in the previous section, the resulting graph does not contain triangular faces and Proposition 8 becomes applicable. We obtain, however, only something we know already: Either one of the three sets percolates, or all three of them form rings; by invoking symmetry we conclude that either the equatorial strip percolates or all three sets form rings.

More interesting is what can be said for the circle $S_1$. We can decompose it into a number of disjoint intervals and the resulting graph will be that of $\mathbb{Z}_n$. Choosing $n=4$, we obtain the following result:

**Proposition 9.** Let $f$: $T \to S_1$ be a Lipschitz continuous map with sufficiently small Lipschitz constant $\varepsilon$ (it suffices to choose $\varepsilon < \sqrt{2}$). Then either there is a half-circle that percolates or there is an interval $\mathscr{I} \subset S_1$ of length $|\mathscr{I}| = 2 \arcsin(\varepsilon/2)$ having property (R).

*Proof.* Divide the circle into four intervals: $\mathscr{I}$, its image $\tilde{\mathscr{I}}$ under a rotation by 180°, and two pieces of the complement $\mathscr{J}$ and $\tilde{\mathscr{J}}$. The lengths of these intervals are $|\mathscr{I}|$ and $\pi - |\mathscr{I}|$, respectively. So the Lipschitz continuity does not allow neighboring lattices sites to "jump" across any of the intervals and we are in the situation of the graph $\mathbb{Z}_4$, so Proposition 8 applies, with $V_+$ corresponding to $\mathscr{I} \cup \mathscr{J}$ and $V_-$ corresponding to $\tilde{\mathscr{I}} \cup \tilde{\mathscr{J}}$. So either one of the half-circles corresponding to $V_+$ or $V_-$ percolates, or if that is not the case, one of the four intervals forms rings. But in that case also the adjacent interval in the other half-circle forms rings. ∎

It is easy to see that Proposition 9 can be generalized to some other manifolds: for instance, for any manifold that contains $S_1$ as a factor (such as a torus) or even only contains a factor homotopically equivalent to $S_1$, a similar statement can be proven in a similar way. An example of a manifold that is homotopically equivalent to the circle is given by the "truncated 2-sphere" considered by Richard,[23] $\{S \in S_2 \mid |S_z| < c\}$.

## 6. APPLICATION TO MODELS

We will now apply the topological argument of the previous section to various models of statistical mechanics of the type described in Eq. (1).

First let us consider the $O(2)$ model.

**Theorem 1.** Assume Conjectures 1 and 2. Then the $O(2)$ model on the lattice $T$ with a Lipschitz constraint $\varepsilon < \sqrt{2}$ has no mass gap.

*Proof.* By Proposition 9 either there is a half-circle that percolates or there is an interval $\mathscr{I}$ of length $2 \arcsin(\delta/2)$ with $\varepsilon < \delta < \sqrt{2}$ forming rings. By using the conjectures, we can eliminate the first possibility: according to Conjecture 1, there is a unique translation-invariant Gibbs state which is $O(2)$ invariant. Therefore if a half-circle percolated, so would its complement, in contradiction with Conjecture 2. Hence $\mathscr{I}$ forms rings and again by Conjecture 1 the same is true about any interval of the same length. So neither $\mathscr{I}$ nor its complement percolates, but both form rings. We can regard the union of $\mathscr{I}$ with its image under a rotation by 180° as an equatorial strip and apply Proposition 4, which shows that there is no exponential decay. ∎

As remarked at the end of the previous section, we can apply the same argument to Richard's truncated sphere model with a Lipschitz constraint. Note that Richard's proof, which uses correlation inequalities to relate the model to the $O(2)$ model, cannot be used for our constrained models. For this reason let us state this result separately:

**Theorem 2.** Assume Conjectures 1 and 2. Then the truncated sphere model on the $T$ lattice with a Lipschitz constraint $\varepsilon$ and the spin taking values in $\{S \in S_2 \mid |S_z| < c\}$ has no mass gap provided $\varepsilon < [2(1-c^2)]^{1/2}$.

**Theorem 3.** Assume Conjecture 2. Let $P$ be a polytope in $S_{N-1}$ not having any triangular 2-faces, and on which a finite subgroup $H$ of $O(N)$ acts transitively. Assume furthermore that $H$ contains a reflection given by (12), and that the vertex set of $P$ can be decomposed into two connected subsets $V_+$ and $V_-$ as in (19). Then the models defined by (1) with a Lipschitz constant $\varepsilon$ so small that the spins on neighboring lattice sites are either equal or are connected by an edge of $P$ have either long-range order or nonexponential decay of correlations.

*Proof.* The existence of a reflection symmetry allows the introduction of embedded Ising variables and the FK formalism (as remarked, it does not cause any problems if some of the vertices are invariant under the reflection; the corresponding Ising spins can be assigned arbitrarily since they are not coupled). By the results of the previous section either $V_+$ or $V_-$ percolates or there will be a vertex satisfying (R). In the first case we have spontaneous symmetry breaking by Conjecture 2; in the second we obtain divergent Ising susceptibility, hence nonexponential decay of the Ising correlations. ▮

One may wonder if our results do not imply that at least for small $\beta$ our discrete models are actually massless. But the fact is that even at $\beta = 0$ we cannot prove that there is no long-range order, in spite of the fact that Peierls contours do not cost any energy. We have carried out numerical simulations for the cube which indicate that in fact there exists long-range order even at $\beta = 0$. The reason is of course that in these models Peierls contours cost entropy, and hence free energy, and are suppressed for this reason. A similar phenomenon was discovered by Newman and Stein[(24)] in the low-temperature phase of the Ising-like systems: they show that, contrary to naive expectations, there is an abundance of domains which prefer energetically to be flipped.

Finally, we would like to make a remark on the $\mathbb{Z}$ model, which is a version of the solid-on-solid (SOS) model. We cannot define translation-invariant Gibbs states for this model, but we can try to fix the spin at the origin and take thermodynamic limits. Considering equivalence classes of spin values mod $2k$, we obtain that there is either percolation of a certain spin value or formation of rings for an equivalence class mod $2k$ of spin values. The former happens for large $\beta$, whereas the latter presumably takes place for small $\beta$ (in the "rough" phase).

## APPENDIX. PROOF OF THE MERMIN–WAGNER THEOREM FOR A SIMPLIFIED LIPSCHITZ CONTINUOUS MODEL

The simplified model we are considering in this appendix is inspired by a similar one discussed in Section 6 of ref. 12. It is defined as follows: We start with the n.n. constrained model on the lattice $T$ characterized by the Gibbs factor

$$\tilde{g}_\beta(S \cdot S') = \exp[-\beta h(S \cdot S')]\, \theta(S \cdot S' - c) \tag{A.1}$$

for each n.n. pair $x, y$ with spins $S$, $S'$; $c = 1 - \varepsilon^2/2$, $\theta$ is the Heaviside function, and $h \in C^2$. It is assumed that $h$ satisfies the following conditions:

$$\begin{aligned} &h(1) = 0 \\ &h'(1) = -1 \\ &h'(0) \leqslant 0 \qquad \text{for} \quad x \in [0, 1] \end{aligned} \tag{A.2}$$

[the normalizations of $h(1)$ and $h'(1)$ clearly do not imply any loss of generality].

Now consider the sequence $L_0, L_1, L_2,...$ of concentric hexagonal loops on the dual lattice surrounding the origin in $T$.

We send $\beta \to \infty$ on all bonds of $T$ that do not cross one of the loops $L_0, L_1, L_2,...$. In other words, we force all spins connected by those bonds to be equal. It is intuitively clear that this makes the model more ordered, even though no correlation inequalities are known to us that would make this into a proven mathematical statement.

The resulting model can be considered as a half-infinite chain of $O(N)$ spins

$$S_1, S_2, S_3,... \tag{A.3}$$

coupled through the non-translation-invariant Gibbs factors

$$g_{k,k+1}(S_k \cdot S_{k+1}) = z_k(\beta)^{-1} \exp[-K_k \beta h(S_k \cdot S_{k+1})]\, \theta(S_k \cdot S_{k+1}) \tag{A.4}$$

where the normalization $z_k(\beta)$ is chosen such that $\int g_{k,k+1}(S \cdot S')\, dS' = 1$ and $K_k = 12k + 6$. The Gibbs factors (A.4) can be interpreted as "radial transfer matrices" of the model.

We claim that the model does not have long-range order at any $\beta$; more specifically, we will prove the following result:

**Proposition A.1.** The $O(N)$ model defined by (A.3) has a unique $O(N)$ invariant "Gibbs" state.

*Proof.* If suffices to show that the probability distribution of $S_1, S_2,..., S_r$ given $S_k$, $S_{k+1}$, $S_{k+3},...$ converges weakly to a unique $O(N)$ invariant measure as $k \to \infty$. We will show actually that for $k \to \infty$

$$p(dS_1,..., dS_r \mid S_k, S_{k+1},...) \to \prod_{j=1}^{r-1} g_{j,j+1}(S_j \cdot S_{j+1}) \prod_{j=1}^{r} dS_j \tag{A.5}$$

By the one-step Markov property of our model for $k > r$ the left-hand side of (A.5) can be rewritten simply as

$$p(dS_1,..., dS_r \mid S_k, S_{k+1},...) = p(dS_1,..., dS_r \mid S_k) \tag{A.6}$$

and it is given by

$$p(dS_1,..., dS_r \mid S_k) = \prod_{j=1}^{r} dS_j \int \prod_{i=1}^{k} g_{i,i+1}(S_i \cdot S_{i+1}) \prod_{i=r+1}^{k-1} dS_i \tag{A.7}$$

We now expand the Gibbs factors in Gegenbauer polynomials, corresponding to the irreducible representations of $O(N)$:

$$g_{k,k+1}(t) = \sum_{l=0}^{\infty} a_l^{(k)}(\beta)\, \hat{C}_l^{\nu}(t) \tag{A.8}$$

where $\hat{C}_l^\nu(t) = C_l^\nu(t)\, C_l^\nu(1)/h_l$, $C_l^\nu$ is the standard Gegenbauer polynomial (see, for instance, ref. 25), $h_l = \int dS\, [C_l^\nu(S \cdot S_o)]^2$, and $\nu = (N-2)/2$. The expansion (A.8) is at least convergent in the sense of $L^2([-1, 1], (1-t^2)^{\nu - 1/2}\, dt)$.

With the chosen normalization the $\hat{C}_l^\nu$ are the kernels of orthogonal projections:

$$\int dS\, \hat{C}_l^\nu(S_1 \cdot S)\, \hat{C}_k^\nu(S \cdot S_2) = \delta_{lk} \hat{C}_l^\nu(S_1 \cdot S_2) \tag{A.9}$$

as follows, for instance, from the addition theorem for the Gegenbauer polynomials [Eq. (34) on p. 178 of ref. 25].

Therefore the "convolutions" in (A.7) correspond simply to multiplication of the expansion coefficients:

$$\begin{aligned}\int dS_k\, g_{k-1,k}(S_{k-1} \cdot S_k)\, g_{k,k+1}(S_k \cdot S_{k+1})\\ = \sum_{l=0}^{\infty} \hat{C}_l^\nu(S_{k-1} \cdot S_{k+1})\, a_l^{(k-1)}(\beta)\, a_l^{(k)}(\beta)\end{aligned} \tag{A.10}$$

(A.5) and hence Proposition A.1 will follow from the next result:

**Lemma A.1.** For $l \neq 0$, $\lim_{k\to\infty} \prod_{j=r}^{k-1} a_l^{(j)} = 0$, whereas for $l = 0$, it is equal to 1.

*Proof.* The second statement is a trivial consequence of the normalization condition of the Gibbs factors $g_{k,k+1}$.

To see the first statement, note that the coefficients $a_l^{(j)}$ are given by

$$a_l^{(j)}(\beta) = \zeta_j^{-1} \int_c^1 \exp[-K_j \beta h(t)]\, \tilde{C}_l^\nu(t)(1-t^2)^{\nu - 1/2}\, dt \tag{A.11}$$

where $\tilde{C}_l^\nu(t) = C_l^\nu(t)/C_l^\nu(1)$ and $\zeta_j(\beta)$ is chosen such that $a_l^{(j)} = 1$. Therefore $a_l^{(j)}(\beta) = b_l(K_j \beta)$, with

$$b_l(\gamma) = \zeta_1(\gamma)^{-1} \int_c^1 \exp[-\gamma h(t)]\, \tilde{C}_l^\nu(t)(1-t^2)^{\nu - 1/2}\, dt \tag{A.12}$$

We claim now the following result:

**Lemma A.2.** For $\gamma \to \infty$

$$b_l(\gamma) = 1 - \frac{c_l}{2\gamma} + O\left(\frac{1}{\gamma^2}\right) \tag{A.13}$$

with $c_l = l(l + 2\nu)$.

*Remark.* $c_l$ is the value of the quadratic Casimir operator of $O(N)$ on the eigenspace corresponding to the "angular momentum" $l$ or equivalently the eigenvalue of the Laplace–Beltrami operator $\Delta_{\mathrm{LB}}$ on that space.

*Proof of Lemma A.2.* We use the Taylor expansion of $h(t)$ at $t=1$ to write

$$h(1-x)=x+\frac{b}{2}x^2+r(x) \tag{A.14}$$

where $r(x)=o(x^2)$. So we obtain

$$b_l(\gamma)=\zeta_1(\gamma)^{-1}\int_0^c \exp\left[-\gamma x-\frac{\gamma b}{2}x^2-\gamma r(x)\right]\tilde{C}_l^\nu(1-x)(2x-x^2)^{\nu-1/2}\,dx \tag{A.15}$$

Let us consider the case $b=r=0$. Then

$$b_l(\gamma)=\zeta_1(\gamma)^{-1}\int_0^c e^{-\gamma x}\tilde{C}_l^\nu(1-x)(2x-x^2)^{\nu-1/2}\,dx \tag{A.16}$$

To evaluate (A.16), consider

$$\int_0^c e^{-\gamma x}x^{p-1}\,dx=\gamma^{-p}\int_0^{\gamma c}e^{-y}y^{p-1}\,dy=\gamma^{-p}\Gamma(p)+O(e^{-\gamma c}) \tag{A.17}$$

Therefore

$$\zeta_1(\gamma)=2^{\nu-1/2}\gamma^{-\nu-1/2}\left(\nu+\frac{1}{2}\right)\left[1-\frac{\nu^2-1/4}{2\gamma}+O(\gamma^{-2})\right] \tag{A.18}$$

The leading behavior of the numerator of (A.16) can be determined similarly; the result is

$$b_l(\gamma)=1-\frac{\nu+1/2}{\gamma}\frac{d}{dt}\tilde{C}_l^\nu(t)\bigg|_{t=1}+O(\gamma^{-2}) \tag{A.19}$$

The constant in (A.19) can be found easily from the generating function of the Gegenbauer polynomials:

$$(1-2zt+z^2)^{-\nu}=\sum_{l=0}^{\infty}C_l^\nu(t)z^l \tag{A.20}$$

and the result is $(d/dt)\,\tilde{C}_l^\nu(t)|_{t=1}=c_l/(2\nu+1)$, so for the special case $b=r=0$ Lemma A.2 is proven.

Returning to the general case, we see that the normalization $\zeta_1(\gamma)$ as well as the numerator of (A.12) will only change to $O(\gamma^{-2})$ relative to the leading term. Therefore $b_l(\gamma)$ will change only by a correction of order $\gamma^{-2}$, so that Lemma A.2 is also proven in general. ▮

We can now complete the proof of Lemma A.1:

$$\prod_{j=r}^{k-1} a_l^{(j)}(\beta) = \prod_{j=r}^{k-1} b_l(K_j\beta) = \prod_{j=r}^{k-1} \left[1 - \frac{c_l}{K_j\beta} + r_l(j)\right] \tag{A.21}$$

with $r_l(j) = O(j^{-2})$. It is now easy to see that the right-hand side of (A.21) goes to zero as $k \to \infty$: Taking the logarithm, we obtain

$$\begin{aligned}\sum_{j=r}^{k-1} &\left[\ln\left(1 - \frac{c_l}{K_j\beta}\right) + \ln\left(1 + \frac{r_j}{1 - c_l/K_j\beta}\right)\right] \\ &\leqslant \sum_{j=r}^{k-1} \left(\frac{-c_l}{K_j\beta} + \frac{r_j}{1 - c_l/4\beta}\right)\end{aligned} \tag{A.22}$$

The second term is bounded as $k \to \infty$, whereas the first term goes logarithmically to $-\infty$, so Lemma A.1 is proven. ▮

Since our simplified model has less fluctuations than the full model defined by (1), we have not doubt that the Mermin–Wagner theorem also holds for the latter and Conjecture 1 is true, even though it may be hard to construct a proof. Our result also shows that the $C^2$ condition that is usually imposed on the Hamiltonian is too strong; it should be enough if it holds for small $|S(x) - S(y)|$. The reason is that the large fluctuations at large distances that are underlying the Mermin–Wagner theorem can be built up from small local fluctuations.

## ACKNOWLEDGMENTS

We are grateful to the Max-Planck-Institut für Physik and the Center for the Study of Complex Systems of the University of Arizona for their continued support of the research reported here.

## REFERENCES

1. P. W. Kasteleyn and C. M. Fortuin, *J. Phys. Soc. Jpn.* **26**(Suppl.):11 (1969); C. M. Fortuin and P. W. Kasteleyn, *Physica* (Utrecht) **57**:536 (1972).
2. R. C. Brower and P. Tamayo, *Phys. Rev. Lett.* **62**:1087 (1989).
3. A. Patrascioiu, *Nuovo Cimento* **108B**:91 (1989).
4. J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **81**:527 (1981).

5. A. Patrascioiu, Existence of algebraic decay in non-Abelian ferromagnets, University of Arizona preprint (1991).
6. T. E. Harris, *Proc. Camb. Phil. Soc.* **43**:13 (1960).
7. A. Patrascioiu and E. Seiler, Applying percolation techniques to two-dimensional ferromagnets, University of Arizona preprint (1991) and Max-Planck-Institut für Physik preprint MPI-Ph/91-94.
8. H. O. Georgii, *Commun. Math. Phys.* **81**:455 (1981).
9. J. Bricmont and A. Fontaine, *J. Stat. Phys.* **26**:745 (1981).
10. H. O. Georgii, *Gibbs Measures and Phase Transitions* (de Gruyter, Berlin, 1988).
11. N. D. Mermin, *J. Math. Phys.* **8**:1061 (1967).
12. R. L. Dobrushin and S. B. Shlosman, *Commun. Math. Phys.* **42**:31 (1975).
13. C. Pfister, *Commun. Math. Phys.* **79**:181 (1981).
14. J. Fröhlich and C. Pfister, *Commun. Math. Phys.* **81**:277 (1981).
15. J. Bricmont, J. R. Fontaine, and L. J. Landau, *Commun. Math. Phys.* **56**:281 (1977).
16. A. Messager, S. Miracle-Sole, and C. Pfister, *Commun. Math. Phys.* **58**:19 (1978).
17. M. Miyamoto, *Commun. Math. Phys.* **44**:169 (1975).
18. A. Coniglio, C. R. Nappi, F. Peruggi, and L. Russo, *Commun. Math. Phys.* **51**:315 (1976).
19. L. Russo, *Z. Wahrsch. Verw. Gebiete* **43**:39 (1975).
20. F. Niedermayer, *Phys. Rev. Lett.* **61**:2026 (1988).
21. U. Wolff, *Phys. Rev. Lett.* **62**:361 (1989).
22. M. Hasenbusch, *Nucl. Phys. B* **333**:581 (1990).
23. J.-L. Richard, *Phys. Lett.* **184B**:75 (1987).
24. C. Newman and D. Stein, *Phys. Rev. Lett.* **65**:460 (1990).
25. A. Erdélyi *et al.*, *Higher Transcendental Functions*, Vol. 2 (McGraw-Hill, New York, 1953).


822/69/3-4-10